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COEFFICIENT CONDITIONS FOR HARMONIC CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. New sufficient conditions, concerned with the coefficients of harmonic functions $f(z) = h(z) + \overline{g(z)}$ in the open unit disk \mathbb{U} normalized by $f(0) = h(0) = h'(0) - 1 = 0$, for $f(z)$ to be harmonic close-to-convex functions are discussed. Furthermore, several illustrative examples and the image domains of harmonic close-to-convex functions satisfying the obtained conditions are enumerated.

1. INTRODUCTION

For a continuous complex-valued function $f(z) = u(x, y) + iv(x, y)$ ($z = x + iy$), we say that $f(z)$ is harmonic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ if both $u(x, y)$ and $v(x, y)$ are real harmonic in \mathbb{U} , that is, $u(x, y)$ and $v(x, y)$ satisfy the Laplace's equations

$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{and} \quad \Delta v = v_{xx} + v_{yy} = 0.$$

A complex-valued harmonic function $f(z)$ in \mathbb{U} is given by $f(z) = h(z) + \overline{g(z)}$ where $h(z)$ and $g(z)$ are analytic in \mathbb{U} . We call $h(z)$ and $g(z)$ the analytic part and the co-analytic part of $f(z)$, respectively. A necessary and sufficient condition for $f(z)$ to be locally univalent and sense-preserving in \mathbb{U} is $|h'(z)| > |g'(z)|$ in \mathbb{U} (see, [2] or [8]). Let \mathcal{H} denote the class of harmonic functions $f(z)$ in \mathbb{U} with $f(0) = h(0) = 0$ and $h'(0) = 1$. Thus, every normalized harmonic function $f(z)$ can be written by

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in \mathcal{H}$$

where $a_1 = 1$ and $b_0 = 0$, for convenience.

We next denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f(z) \in \mathcal{H}$ which are univalent and sense-preserving in \mathbb{U} . Since the sense-preserving property of $f(z)$, we see that $|b_1| = |g'(0)| < |h'(0)| = 1$. If $g(z) \equiv 0$, then $\mathcal{S}_{\mathcal{H}}$ reduces to the class \mathcal{S} consisting of normalized analytic univalent functions. Furthermore, for every function $f(z) \in \mathcal{S}_{\mathcal{H}}$, the function

$$F(z) = \frac{f(z) - \overline{b_1 f(z)}}{1 - |b_1|^2} = z + \sum_{n=2}^{\infty} \frac{a_n - \overline{b_1} b_n}{1 - |b_1|^2} z^n + \overline{\sum_{n=2}^{\infty} \frac{b_n - b_1 a_n}{1 - |b_1|^2} z^n}$$

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is also a member of $\mathcal{S}_{\mathcal{H}}$. Therefore, we consider the subclass $\mathcal{S}_{\mathcal{H}}^0$ of $\mathcal{S}_{\mathcal{H}}$ defined as

$$\mathcal{S}_{\mathcal{H}}^0 = \{f(z) \in \mathcal{S}_{\mathcal{H}} : b_1 = g'(0) = 0\}.$$

Conversely, if $F(z) \in \mathcal{S}_{\mathcal{H}}^0$, then $f(z) = F(z) + \overline{b_1 F(z)} \in \mathcal{S}_{\mathcal{H}}$ for any b_1 ($|b_1| < 1$).

We say that a domain \mathbb{D} is a close-to-convex domain if the complement of \mathbb{D} can be written as a union of non-intersecting half-lines (except that the origin of one half-line may lie on one of the other half-lines). Let \mathcal{C} , $\mathcal{C}_{\mathcal{H}}$ and $\mathcal{C}_{\mathcal{H}}^0$ be the respective subclasses of \mathcal{S} , $\mathcal{S}_{\mathcal{H}}$ and $\mathcal{S}_{\mathcal{H}}^0$ consisting of all functions $f(z)$ which map \mathbb{U} onto a certain close-to-convex domain.

Bshouty and Lyzzaik [1] have stated the following result.

Theorem 1.1. *If $f(z) = h(z) + \overline{g(z)} \in \mathcal{H}$ satisfies*

$$g'(z) = zh'(z) \quad \text{and} \quad \operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}$$

for all $z \in \mathbb{U}$, then $f(z) \in \mathcal{C}_{\mathcal{H}}^0 \subset \mathcal{S}_{\mathcal{H}}^0$.

A simple and interesting example is below.

Example 1.1. *The function*

$$f(z) = \frac{1 - (1 - z)^2}{2(1 - z)^2} + \frac{\overline{z^2}}{2(1 - z)^2} = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n + \sum_{n=2}^{\infty} \frac{n-1}{2} \overline{z}^n$$

satisfies the conditions of Theorem 1.1, and therefore $f(z)$ belongs to the class $\mathcal{C}_{\mathcal{H}}^0$. We now show that $f(\mathbb{U})$ is actually close-to-convex domain. It follows that

$$\begin{aligned} f(z) &= \left(\frac{z}{2(1-z)^2} + \frac{z}{2(1-z)} \right) + \overline{\left(\frac{z}{2(1-z)^2} - \frac{z}{2(1-z)} \right)} \\ &= \operatorname{Re} \left(\frac{z}{(1-z)^2} \right) + i \operatorname{Im} \left(\frac{z}{1-z} \right). \end{aligned}$$

Setting

$$f(re^{i\theta}) = \frac{-2r^2 + r(1+r^2)\cos\theta}{(1+r^2-2r\cos\theta)^2} + \frac{r\sin\theta}{1+r^2-2r\cos\theta}i = u + iv$$

for any $z = re^{i\theta} \in \mathbb{U}$ ($0 \leq r < 1$, $0 \leq \theta < 2\pi$), we see that

$$-4(u + v^2) = \frac{4r(r - \cos\theta)(1 - r\cos\theta)}{(1 + r^2 - 2r\cos\theta)^2} = \frac{4r(r - t)(1 - rt)}{(1 + r^2 - 2rt)^2} \equiv \phi(t) \quad (-1 \leq t = \cos\theta \leq 1).$$

Since

$$\phi'(t) = \frac{-4r(1-r^2)^2}{(1+r^2-2rt)^3} \leq 0,$$

we obtain that

$$\phi(t) \leq \phi(-1) = \frac{4r}{(1+r)^2} \equiv \psi(r).$$

Also, noting that

$$\psi'(r) = \frac{4(1-r)}{(1+r)^3} > 0,$$

we know that

$$\psi(r) < \psi(1) = 1$$

which implies that

$$u > -v^2 - \frac{1}{4}.$$

Thus, $f(z)$ maps \mathbb{U} onto the following close-to-convex domain.

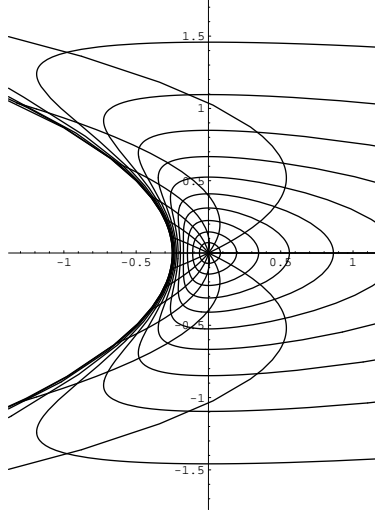


FIGURE 1. The image of $f(z) = \frac{1 - (1-z)^2}{2(1-z)^2} + \frac{z^2}{2(1-z)^2}$.

Remark 1.1. Let \mathcal{M} be the class of all functions satisfying the conditions of Theorem 1.1. Then, it was earlier conjectured by Mocanu [9, 10] that $\mathcal{M} \subset \mathcal{S}_{\mathcal{H}}^0$. Furthermore, we can immediately see that the function $f(z)$ in Example 1.1 is a member of the class \mathcal{M} and it shows that $f(z) \in \mathcal{M}$ is not necessarily starlike with respect to the origin in \mathbb{U} ($f(z)$ is starlike with respect to the origin in \mathbb{U} if and only if $tw \in f(\mathbb{U})$ for all $w \in f(\mathbb{U})$ and t ($0 \leq t \leq 1$)).

Remark 1.2. For the function $f(z) = h(z) + \overline{g(z)} \in \mathcal{H}$ given by

$$g'(z) = z^{n-1}h'(z) \quad (n = 2, 3, 4, \dots),$$

letting $w(t) = f(e^{it}) = h(e^{it}) + \overline{g(e^{it})}$ ($-\pi \leq t < \pi$), we know that

$$\operatorname{Im} \left(\frac{w''(t)}{w'(t)} \right) \leq 0 \quad (-\pi \leq t < \pi)$$

which means that $f(z)$ maps the unit circle $\partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ onto a union of several concave curves (see, [6, Theorem 2.1]).

Jahangiri and Silverman [7] have given the following coefficient inequality for $f(z) \in \mathcal{H}$ to be in the class $\mathcal{C}_{\mathcal{H}}$.

Theorem 1.2. *If $f(z) \in \mathcal{H}$ satisfies*

$$\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1,$$

then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Example 1.2. *The function*

$$f(z) = z + \frac{1}{5}\bar{z}^5$$

belongs to the class $\mathcal{C}_{\mathcal{H}}^0 \subset \mathcal{C}_{\mathcal{H}}$ and satisfies the condition of Theorem 1.2. Indeed, $f(z)$ maps \mathbb{U} onto the following hypocycloid of six cusps (cf. [3] or [6]).

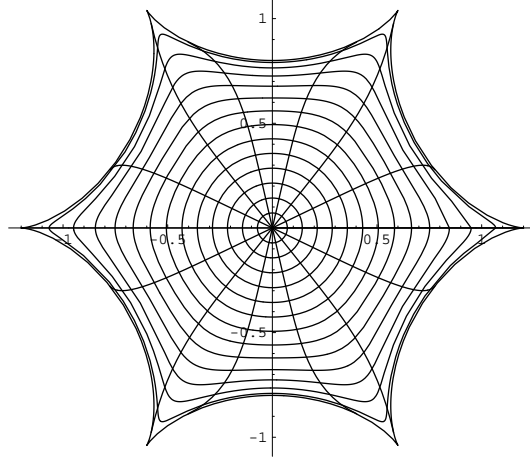


FIGURE 2. The image of $f(z) = z + \frac{1}{5}\bar{z}^5$.

The object of this paper is to find some sufficient conditions for functions $f(z) \in \mathcal{H}$ to be in the class $\mathcal{C}_{\mathcal{H}}$. In order to establish our results, we have to recall here the following lemmas due to Clunie and Sheil-small [2].

Lemma 1.1. *If $h(z)$ and $g(z)$ are analytic in \mathbb{U} with $|h'(0)| > |g'(0)|$ and $h(z) + \varepsilon g(z)$ is close-to-convex for each ε ($|\varepsilon| = 1$), then $f(z) = h(z) + \overline{g(z)}$ is harmonic close-to-convex.*

Lemma 1.2. *If $f(z) = h(z) + \overline{g(z)}$ is locally univalent in \mathbb{U} and $h(z) + \varepsilon g(z)$ is convex for some ε ($|\varepsilon| \leq 1$), then $f(z)$ is univalent close-to-convex.*

We also need the following result due to Hayami, Owa and Srivastava [5].

Lemma 1.3. *If a function $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ is analytic in \mathbb{U} and satisfies*

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} j(j+1) \binom{\alpha}{k-j} A_j \right\} \binom{\beta}{n-k} \right| \right. \\ \left. + \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} j(j-1) \binom{\alpha}{k-j} A_j \right\} \binom{\beta}{n-k} \right| \right] \leq 2$$

for some real numbers α and β , then $F(z)$ is convex in \mathbb{U} .

2. MAIN RESULTS

Our first result is contained in

Theorem 2.1. *If $f(z) \in \mathcal{H}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} |na_n - e^{i\varphi}(n-1)a_{n-1}| + \sum_{n=1}^{\infty} |nb_n - e^{i\varphi}(n-1)b_{n-1}| \leq 1$$

for some real number φ ($0 \leq \varphi < 2\pi$), then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Proof. Let $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ be analytic in \mathbb{U} . If $F(z)$ satisfies

$$\sum_{n=2}^{\infty} |nA_n - e^{i\varphi}(n-1)A_{n-1}| \leq 1$$

then it follows that

$$\begin{aligned} |(1 - e^{i\varphi}z)F'(z) - 1| &= \left| \sum_{n=2}^{\infty} (nA_n - e^{i\varphi}(n-1)A_{n-1}) z^{n-1} \right| \\ &\leq \sum_{n=2}^{\infty} |nA_n - e^{i\varphi}(n-1)A_{n-1}| \cdot |z|^{n-1} \\ &< \sum_{n=2}^{\infty} |nA_n - e^{i\varphi}(n-1)A_{n-1}| \leq 1 \quad (z \in \mathbb{U}). \end{aligned}$$

This gives us that

$$\operatorname{Re}((1 - e^{i\varphi}z)F'(z)) > 0 \quad (z \in \mathbb{U}),$$

that is, that $F(z) \in \mathcal{C}$. Then, it is sufficient to prove that

$$F(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b_1} = z + \sum_{n=2}^{\infty} \frac{a_n + \varepsilon b_n}{1 + \varepsilon b_1} z^n \in \mathcal{C}$$

for each ε ($|\varepsilon| = 1$) by Lemma 1.1. From the assumption of the theorem, we obtain that

$$\begin{aligned} \sum_{n=2}^{\infty} \left| n \frac{a_n + \varepsilon b_n}{1 + \varepsilon b_1} - e^{i\varphi}(n-1) \frac{a_{n-1} + \varepsilon b_{n-1}}{1 + \varepsilon b_1} \right| \\ \leq \frac{1}{1 - |b_1|} \sum_{n=2}^{\infty} [|na_n - e^{i\varphi}(n-1)a_{n-1}| + |nb_n - e^{i\varphi}(n-1)b_{n-1}|] \leq \frac{1 - |b_1|}{1 - |b_1|} = 1. \end{aligned}$$

This completes the proof of the theorem. \square

Example 2.1. *The function*

$$f(z) = -\log(1-z) + \overline{(-mz - \log(1-z))} = z + \sum_{n=2}^{\infty} \frac{1}{n} z^n + (1-m)\bar{z} + \sum_{n=2}^{\infty} \frac{1}{n} \bar{z}^n \quad (0 < m \leq 1)$$

satisfies the condition of Theorem 2.1 with $\varphi = 0$ and belongs to the class $\mathcal{C}_{\mathcal{H}}$. In particular, putting $m = 1$, we obtain the following.

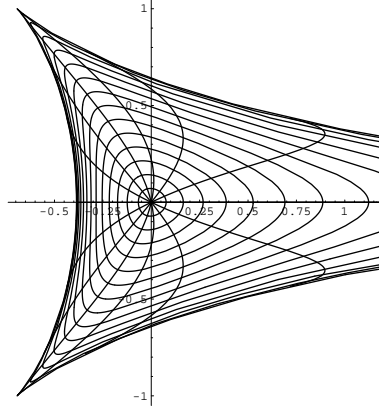


FIGURE 3. The image of $f(z) = -\bar{z} - 2 \log|1-z|$.

By making use of Lemma 1.2 with $\varepsilon = 0$ and applying Lemma 1.3, we readily obtain the next theorem.

Theorem 2.2. *If $f(z) \in \mathcal{H}$ is locally univalent in \mathbb{U} and satisfies*

$$\begin{aligned} \sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} j(j+1) \binom{\alpha}{k-j} a_j \right\} \binom{\beta}{n-k} \right| \right. \\ \left. + \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} j(j-1) \binom{\alpha}{k-j} a_j \right\} \binom{\beta}{n-k} \right| \right] \leq 2 \end{aligned}$$

for some real numbers α and β , then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Putting $\alpha = \beta = 0$ in the above theorem, we arrive at the following result due to Jahangiri and Silverman [7].

Theorem 2.3. *If $f(z) \in \mathcal{H}$ is locally univalent in \mathbb{U} with*

$$\sum_{n=2}^{\infty} n^2 |a_n| \leq 1,$$

then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Furthermore, taking $\alpha = 1$ and $\beta = 0$ in the theorem, we have

Corollary 2.1. *If $f(z) \in \mathcal{H}$ is locally univalent in \mathbb{U} and satisfies*

$$\sum_{n=2}^{\infty} \{n |(n+1)a_n - (n-1)a_{n-1}| + (n-1) |na_n - (n-2)a_{n-1}|\} \leq 2,$$

then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Example 2.2. *The function*

$$f(z) = - \int_0^z \frac{\log(1-t)}{t} dt + \overline{\left(z + (1-z) \log(1-z) \right)} = z + \sum_{n=2}^{\infty} \frac{1}{n^2} z^n + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \bar{z}^n$$

satisfies the conditions of Corollary 2.1 and belongs to the class $\mathcal{C}_{\mathcal{H}}$.

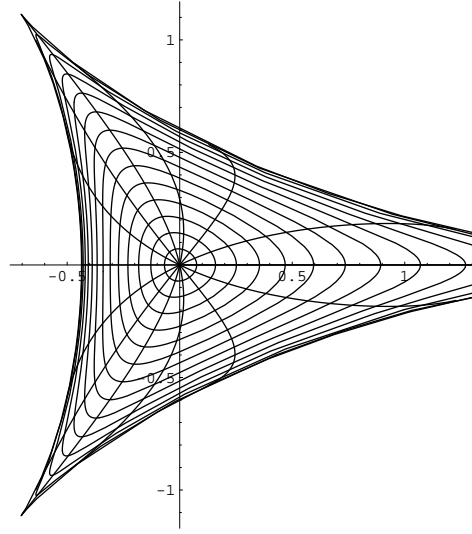


FIGURE 4. The image of $f(z) = - \int_0^z \frac{\log(1-t)}{t} dt + \overline{\left(z + (1-z) \log(1-z) \right)}$.

3. APPENDIX

A sequence $\{c_n\}_{n=0}^{\infty}$ of non-negative real numbers is called a convex null sequence if $c_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$c_n - c_{n+1} \geq c_{n+1} - c_{n+2} \geq 0$$

for all n ($n = 0, 1, 2, \dots$).

The next lemma was obtained by Fejér [4].

Lemma 3.1. *Let $\{c_n\}_{n=0}^{\infty}$ be a convex null sequence. Then, the function*

$$p(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n$$

is analytic and satisfies $\operatorname{Re}(p(z)) > 0$ in \mathbb{U} .

Applying the above lemma, we deduce

Theorem 3.1. *For some b ($|b| < 1$) and some convex null sequence $\{c_n\}_{n=0}^{\infty}$ with $c_0 = 2$, the function*

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} \frac{c_{n-1}}{n} z^n + b \overline{\left(z + \sum_{n=2}^{\infty} \frac{c_{n-1}}{n} z^n \right)}$$

belongs to the class $\mathcal{C}_{\mathcal{H}}$.

Proof. Let us define $F(z)$ by

$$F(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b} = z + \sum_{n=2}^{\infty} \frac{c_{n-1}}{n} z^n$$

for each ε ($|\varepsilon| = 1$). Then, we know that

$$F'(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n \quad (c_0 = 2).$$

By virtue of Lemma 1.1 and Lemma 3.1, it follows that $\operatorname{Re}(F'(z)) > 0$ ($z \in \mathbb{U}$), that is, $F(z) \in \mathcal{C}$. Thus, we conclude that $f(z) = h(z) + \overline{g(z)} \in \mathcal{C}_{\mathcal{H}}$. \square

In the same manner, we also have

Theorem 3.2. *For some b ($|b| < 1$) and some convex null sequence $\{c_n\}_{n=0}^{\infty}$ with $c_0 = 2$, the function*

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} c_j \right) z^n + b \overline{\left(z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} c_j \right) z^n \right)}$$

belongs to the class $\mathcal{C}_{\mathcal{H}}$.

Proof. Let us define $F(z)$ by

$$F(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b} = z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} c_j \right) z^n$$

for each ε ($|\varepsilon| = 1$). Then, we know that

$$(1 - z)F'(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n \quad (c_0 = 2).$$

Therefore, by the help of Lemma 1.1 and Lemma 3.1, we obtain that $\operatorname{Re}((1 - z)F'(z)) > 0$ ($z \in \mathbb{U}$), that is, $F(z) \in \mathcal{C}$ which implies that $f(z) = h(z) + \overline{g(z)} \in \mathcal{C}_{\mathcal{H}}$. \square

Remark 3.1. The sequence

$$\{c_n\}_{n=0}^{\infty} = \left\{ 2, 1, \frac{2}{3}, \dots, \frac{2}{n+1}, \dots \right\}$$

is a convex null sequence because

$$\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} \left(\frac{2}{n+1} \right) = 0, \quad c_n - c_{n+1} = \frac{2}{(n+1)(n+2)} \geq 0$$

and

$$(c_n - c_{n+1}) - (c_{n+1} - c_{n+2}) = \frac{4}{(n+1)(n+2)(n+3)} \geq 0 \quad (n = 0, 1, 2, \dots).$$

Setting $b = \frac{1}{4}$ in Theorem 3.1 with the above sequence $\{c_n\}_{n=0}^{\infty}$, we derive

Example 3.1. *The function*

$$f(z) = -z - 2 \int_0^z \frac{\log(1-t)}{t} dt - \frac{1}{4} \overline{\left(z + 2 \int_0^z \frac{\log(1-t)}{t} dt \right)} = z + \sum_{n=2}^{\infty} \frac{2}{n^2} z^n + \frac{1}{4} \overline{\left(z + \sum_{n=2}^{\infty} \frac{2}{n^2} z^n \right)}$$

is in the class $\mathcal{C}_{\mathcal{H}}$.

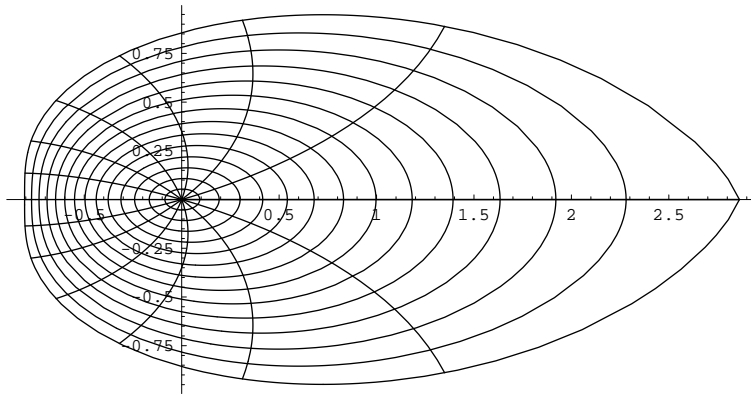


FIGURE 5. The image of $f(z)$ in Example 3.1.

Moreover, we know

Remark 3.2. The sequence

$$\{c_n\}_{n=0}^{\infty} = \left\{ 2, 1, \frac{1}{2}, \dots, 2^{1-n}, \dots \right\}$$

is a convex null sequence because

$$\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} 2^{1-n} = 0, \quad c_n - c_{n+1} = 2^{-n} \geq 0$$

and

$$(c_n - c_{n+1}) - (c_{n+1} - c_{n+2}) = 2^{-(n+1)} \geq 0 \quad (n = 0, 1, 2, \dots).$$

Hence, letting $b = \frac{1}{4}$ in Theorem 3.2 with the sequence $\{c_n\}_{n=0}^{\infty} = \{2^{1-n}\}_{n=0}^{\infty}$, we have

Example 3.2. *The function*

$$\begin{aligned} f(z) &= -3 \log(1-z) + 4 \log\left(1 - \frac{z}{2}\right) + \overline{\left(-\frac{3}{4} \log(1-z) + \log\left(1 - \frac{z}{2}\right)\right)} \\ &= z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} 2^{1-j}\right) z^n + \frac{1}{4} \overline{\left(z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} 2^{1-j}\right) z^n\right)} \end{aligned}$$

is in the class $\mathcal{C}_{\mathcal{H}}$.

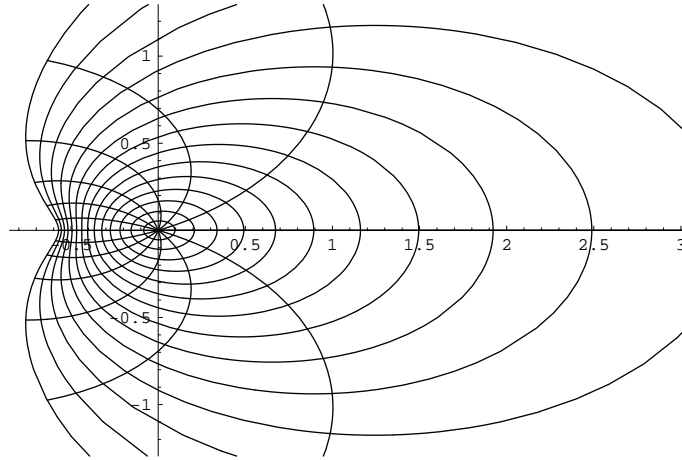


FIGURE 6. The image of $f(z)$ in Example 3.2.

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